Digital Signal Processing 2024S – Assignment 1

Analogue Signals and Systems

Group 52
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Exercise 1 *Complex Numbers*

With $\text{atan2}(y, x) = \arctan\left(\frac{y}{x}\right)$ for $x > 0$ and $\text{atan2}(y, x) = \arctan\left(\frac{y}{x}\right) + \pi$ otherwise.

- $$c_2 = \frac{\sqrt{2}}{2} e^{-\frac{j3\pi}{4}} = \frac{\sqrt{2}}{2} \cos\left(-\frac{3\pi}{4}\right) + j\frac{\sqrt{2}}{2} \sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{2} - j\frac{1}{2}$$
$$c_4 = c_1 + c_2 = -5 - \frac{1}{2} + j\left(3 - \frac{1}{2}\right)$$

- $$c_1 = -5 + j3 = \sqrt{-5^2 + 3^2} \cdot e^{j\,\text{atan2}(3,-5)} = 5.8310 \cdot e^{j2.6012}$$
$$c_5 = c_1 \cdot c_2 = \left(5.8310\frac{\sqrt{2}}{2}\right) e^{j\left(2.6012 + \frac{-3\pi}{4}\right)} = 4.1231 e^{j0.245}$$

- $$c_6 = |c_3|^2 = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}^2 = 1$$

- $$c_7 = \arg(c_3) = \text{atan2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0.7854$$

- TODO
- TODO

Exercise 2 *Fourier Transform*

Using Eulers formula to reformulate the cosine in terms of complex exponentials, we get

$$x(t) = \hat{X}\cos(2\pi f_0 t) = \hat{X}\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} = \underbrace{\frac{\hat{X}}{2} e^{j2\pi f_0 t}}_{x_1(t)} + \overbrace{\frac{\hat{X}}{2} e^{j2\pi(-f_0)t}}^{x_2(t)}$$

Finally, using the given Fourier transform of a complex exponential (p. 38) and the linearity of FT

$$X(f) \overset{\text{linearity}}{=} X_1(f) + X_2(f) \overset{\text{p. 38}}{=} \frac{\hat{X}}{2}\delta(f - f_0) + \frac{\hat{X}}{2}\delta(f + f_0)$$

which is what was to be shown. Figure 1 is a diagram of $X(f)$.

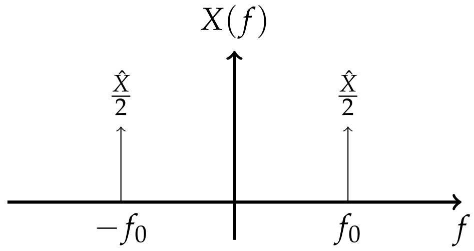

Figure 1: Spectrum of $x(t) = \hat{X}\cos(2\pi f_0 t)$

Exercise 3 *Time Shift and Phase*

a) In general, we can formulate ϕ_i as

$$\begin{aligned} 2\pi f_i t + \phi_i &= 2\pi f_i(t-\tau) \\ \phi_i &= 2\pi f_i t - 2\pi f_i \tau - 2\pi f_i t \\ \phi_i &= -2\pi f_i \tau \end{aligned}$$

And thus for $\tau = 0.1s$ we have $\phi_1 = -0.2\pi$ and $\phi_2 = -\frac{1}{15}\pi$.
We verify that this corresponds to the "shift theorem" by applying it to Y_i and ensuring that the results are as expected.

$$\begin{aligned} X_1(f) &= -\frac{j}{2}\delta(f-f_1) + \frac{j}{2}\delta(f+f_1) \\ Y_1(f) &= \left(-\frac{j}{2}\delta(f-f_1) + \frac{j}{2}\delta(f+f_1)\right) e^{-j2\pi f 0.1} \\ &= -\frac{j}{2}e^{-j0.2\pi f}\delta(f-f_1) + \frac{j}{2}e^{-j0.2\pi f}\delta(f+f_1) \end{aligned}$$

Since $\delta(t)$ is 0 for all $t \neq 0$, only $f = f_1$ and $f = -f_1$ will affect our result. Given $f_1 = 1\text{Hz}$ we can reformulate the above to

$$Y_1(f) = \begin{cases} -\frac{j}{2}e^{-j0.2\pi}\delta(0), & \text{if } f = f_1 \\ \frac{j}{2}e^{j0.2\pi}\delta(0), & \text{if } f = -f_1 \\ 0, & \text{otherwise} \end{cases}$$

where we observe that the exponent matches our calculated ϕ_1.
We can do the same for $Y_2(f)$, where we obtain

$$\begin{aligned} Y_2(f) &= -\frac{j}{2}e^{-j2\pi f 0.1}\delta(f-f_2) + \frac{j}{2}e^{-j2\pi f 0.1}\delta(f+f_2) \\ Y_2(f) &= \begin{cases} -\frac{j}{2}e^{-j\frac{1}{15}\pi}\delta(0), & \text{if } f = f_2 \\ \frac{j}{2}e^{j\frac{1}{15}\pi}\delta(0), & \text{if } f = -f_2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and again see that the exponent matches our calculated ϕ_2.

b) See Figures 2 and 3.

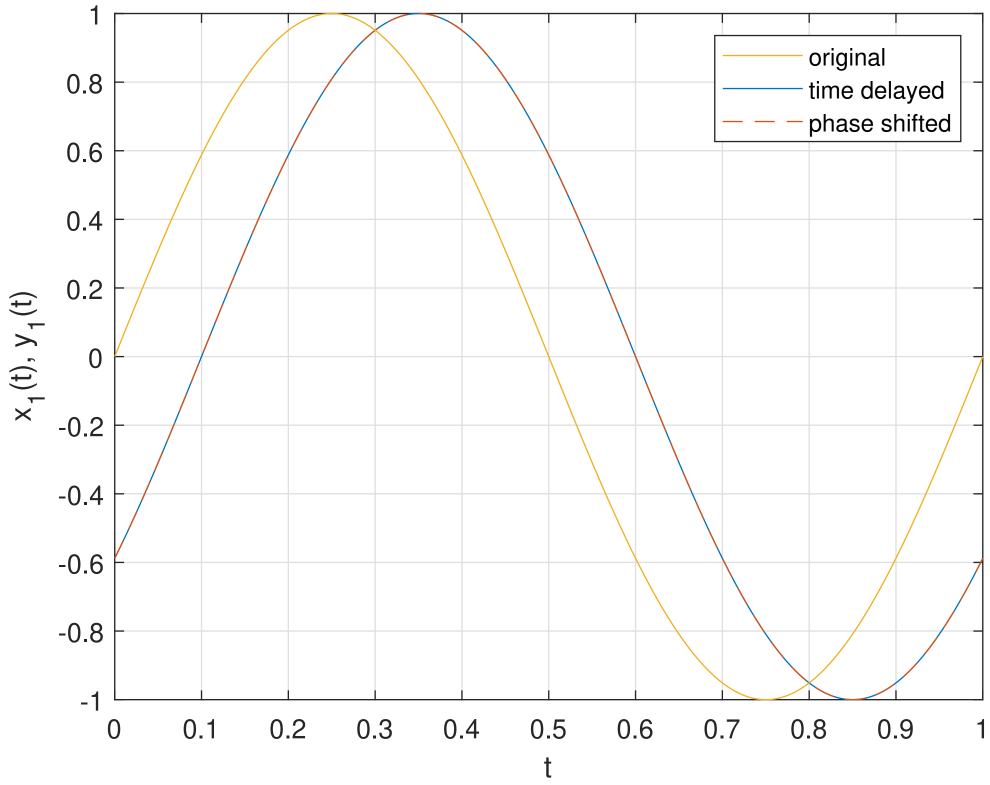

Figure 2: Signals for $f_1 = 1\text{Hz}$

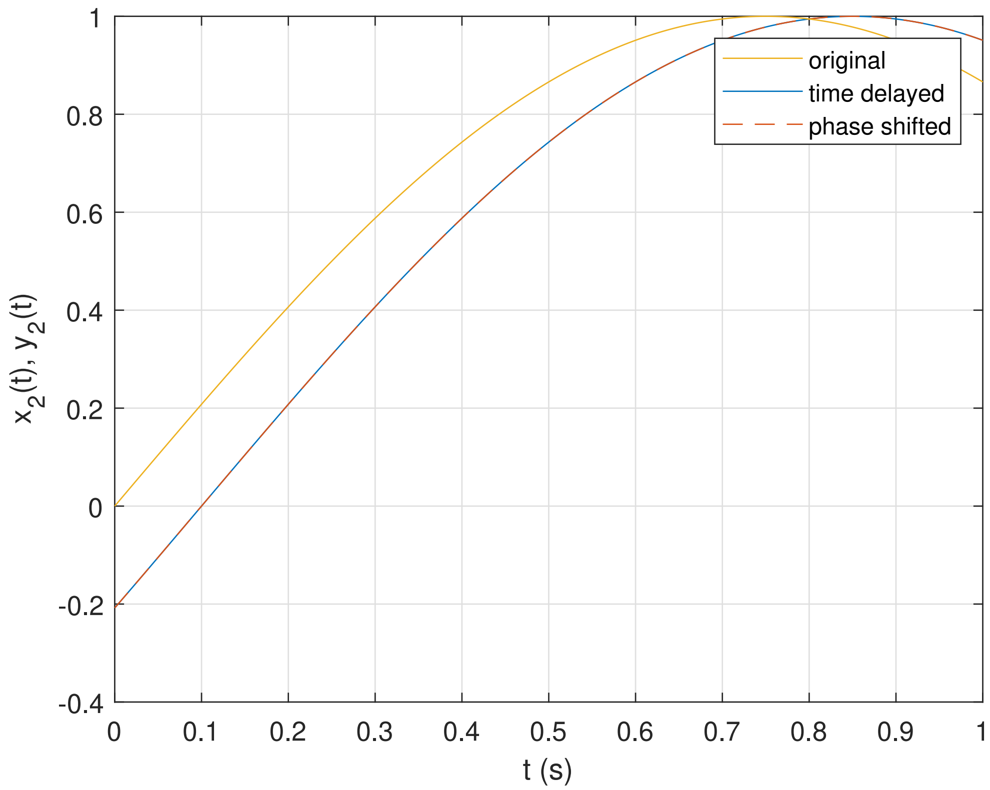

Figure 3: Signals for $f_2 = 3\text{Hz}$

- For arbitrary input signals $x_1(t)$ and $x_2(t)$ with corresponding output signals $y_1(t)$ and $y_2(t)$ let $x(t) = \alpha x_1(t) + \beta x_2(t)$ (α, β arbitrary). Then we have

$$\begin{aligned} y(t) &= (x(t))^2 = (\alpha x_1(t) + \beta x_2(t))^2 = \alpha^2 x_1(t)^2 + 2\alpha\beta x_1(t) x_2(t) + \beta^2 x_2(t)^2 \\ &\neq \alpha y_1(t) + \beta y_2(t) = \alpha (x_1(t))^2 + \beta (x_2(t))^2 \end{aligned}$$

  i.e. the system is not linear.

  Let $x(t)$ be an arbitrary input signal with associated output signal $y(t)$. Let $x'(t)$ be a version of $x(t)$ that is shifted by arbitrary T, $x'(t) = x(t-T)$, with output signal $y'(t)$. Then we have

$$y'(t) = (x'(t))^2 = (x(t-T))^2 = y(t-T)$$

  which demonstrates time-invariance.

- For signals and variables as above, we have

$$\begin{aligned} y(t) &= x(t)\sin(\Omega_0 t) = (x_1(t) + y_1(t))\sin(\Omega_0 t) = x_1(t)\sin(\Omega_0 t) + y_1(t)\sin(\Omega_0 t) \\ &= y_1(t) + y_2(t) \end{aligned}$$

  which establishes linearity. Further, we have

$$y'(t) = x'(t)\sin(\Omega_0 t) = x(t-T)\sin(\Omega_0 t) \neq y(t-T) = x(t-T)\sin(\Omega_0 (t-T))$$

  i.e. the system is not time-invariant.