Time Hierarchy Theorem

Informally, the theorem states that given more time, a Turing machine can solve more problems. (In other words, more languages can be decided with more time.) One way to state it formally is that if f and g are time-honest functions with $f(n) \log f(n) = O(g(n))$, then

$$\mathsf{DTIME}(f(n)) \subsetneq \mathsf{DTIME}(g(n)) .$$

As suggested, we have chosen to present a proof of a weaker version, showing that $\mathsf{DTIME}(n)$ is a strict subset of $\mathsf{DTIME}(n^{1.5})$. The following are notes based on the proof presented in *Computational Complexity: A Modern Approach* by S. Arora and B. Barak, available on p. 66 of the public draft.

Let M_x be the TM represented by a string x with outputs $\{0, 1\}$ and let U be an Universal TM (i.e. an „interpreter“ of TMs, which is itself a TM). Now consider a TM D which, given as input x, runs U for $|x|^{1.4}$ steps, simulating the execution of M_x on x. If an output $M_x(x)$ is obtained in time it outputs $1 - M_x(x)$, the opposite. Otherwise it outputs zero.

Clearly, D finishes execution within $n^{1.4}$ steps since it only lets U run for as many steps and doesn't do anything else that scales with n. Thus the language decided by D is in $\mathsf{DTIME}(n^{1.5})$. The following will show that this language is not in $\mathsf{DTIME}(n)$, thus providing a proof of the initial statement.

For the sake of contradiction assume that there exists a TM M which decides the same language as D but runs in time kn for input sizes n where k is a constant factor independent of n. Then for all possible inputs x we have $M(x) = D(x)$: The machines are functionally equivalent but M runs in linear time.

The time needed to simulate M on U is at most $ck|x| \log |x|$ for some constant c which is independent of x. (This makes use of the general statement that the runtime of an Universal TM is bounded by $cT \log T$ where the simulated machine halts within T steps.) Naturally, at some point $n^{1.4}$ will grow larger than $ckn \log n$. Let n_0 be the smallest n such that this is the case and let x (representing the TM M_x) have a length of n_0. Then $D(x)$ will obtain the output of $M(x)$ within $|x|^{1.4}$ steps (since it stops at that point) but by definition we have

$$D(x) = 1 - M(x) \neq M(x),$$

and have arrived at a contradiction.

Jannik Jungmann (12103135),
Laurenz Weixlbaumer (11804751)