Assume for the sake of contradiction that PALINDROME is regular. Then the pumping lemma must hold for a certain pumping length l. Let $x = 0^l10^l \in$ PALINDROME, then $|x| = 2l + 1 \leq l$. By the pumping lemma we can now decompose this string into substrings u, v, w with $x = uvw$ such that

$$v \neq \epsilon, \qquad |uv| \leq l, \qquad uv^kw \in \text{PALINDROME for all } k \geq 0.$$

Because $uvw = 0^l10^l$, the substring uv can not include any 1s since it can't be long enough to reach the single 1, which is at $l + 1$. So we conclude $v = 0^i$ for some $i \geq 1$ (not ≥ 0 because $i \neq \epsilon$). By the above conditions we can „pump up“ uvw by repeating v an arbitrary amount of times k. Let $k = 2$, then

$$uv^2w = 0^{l+i}10^l$$

must also be in PALINDROME. It clearly isn't since $0^{l+k} \neq 0^l$ or rather $l + k \neq l$ because $k = 2$.