Exercise 1

a) We first determine a base of U by noting that $x = -y - z$ solves $x + y + z = 0$ for arbitrary y and z leading to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y - z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

which shows that $(-1, 1, 0)$ and $(-1, 0, 1)$ form a basis for U (since any element of U can be written as a linear combination of them).

Consider

$$\begin{aligned} \dim(R^3) &= \dim(U) + \dim(R^3/U) \\ 3 &= 2 + \dim(R^3/U) \\ \dim(R^3/U) &= 1 \end{aligned}$$

thus we are looking for one more element in R^3 such that it forms a basis of R^3 alongside our existing vectors.

b) We first determine a base of U. By solving the linear system $x+y+z = 0$ and $x+2y+3z = 0$.

$$\left.\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \qquad \left.\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix}$$

Thus

$$\begin{aligned} y + 2z &= 0 \\ y &= -2z \end{aligned} \qquad \begin{aligned} x + y + z &= 0 \\ x - 2z + z &= 0 \\ x &= z \end{aligned}$$

leading to

$$\begin{pmatrix} x \\ x \\ -2x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

which makes $(1, 1, -2)$ a basis of U.

Exercise 2

a) To show that the given vectors form an orthogonal system it is necessary to show that they are pairwise orthogonal. (That for any vectors $\vec{v}$ and $\vec{u}$ we have $\vec{v} \cdot \vec{u} = 0$) This is the case here.

b) To show that the given vectors form a basis we show that they are linearly independent (skipped, $x_1, \ldots, x_4 = 0$). Since we have four independent vectors of a four-dimensional vector space they form a basis.

c) To determine the coordinates of of $\vec{a}$ in relation to the given basis we solve

0	-10	26	9	6
0	-5	47	-12	37
1	6	66	4	78
2	-3	-33	-2	-39

and get $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 1$, $\lambda_4 = 0$.

Exercise 3

a) (i)

$$x_1^2 + 2x_2^2 \geq 0$$
$$x_1^2 + 2x_2^2 = 0 \Leftrightarrow \vec{x} = \vec{0}$$

(ii)

$$x_1y_1 + 2x_2y_2 = y_1x_1 + 2y_2x_2$$

(iii)

$$(\lambda x_1 + \phi y_1)z_1 + 2(\lambda x_2 + \phi y_2)z_2 = \lambda(x_1z_1 + 2x_2z_2) + \phi(y_1z_1 + 2y_2z_2)$$
$$\lambda x_1z_1 + \phi y_1z_1 + 2(\lambda x_2z_2 + \phi y_2z_2) = \lambda(x_1z_1 + 2x_2z_2) + \phi(y_1z_1 + 2y_2z_2)$$
$$\lambda x_1z_1 + \phi y_1z_1 + 2\lambda x_2z_2 + 2\phi y_zz_2 = \lambda(x_1z_1 + 2x_2z_2) + \phi(y_1z_1 + 2y_2z_2)$$

b) (i) Same as regular scalar product. (No, contradiction.)

(ii)

$$x_1y_2 + x_2y_1 = y_1x_2 + y_2x_1$$

(iii)

$$(\lambda x_1 + \phi y_1)z_2 + (\lambda x_2 + \phi y_2)z_1 = \lambda(x_1z_2 + x_2z_1) + \phi(y_1z_2 + y_2z_1)$$
$$\lambda x_1z_2 + \phi y_1z_2 + \lambda x_2z_1 + \phi y_2z_1 = \lambda(x_1z_2 + x_2z_1) + \phi(y_1z_2 + y_2z_1)$$

c) Not a scalar product.

(i) Same as regular scalar product.

(ii)

$$x_1y_1 + x_2y_1 = y_1x_1 + y_2x_1$$
$$x_2y_1 = y_2x_1$$

Consider $\vec{x} = (1, 2)$ and $\vec{y} = (3, 4)$, we now have $6 = 4$.

d) Not a scalar product.

(i)

$$x_1 + x_2 + x_1 + x_2 \geq 0$$

Consider $\vec{x} = (-1, 0)$ we now have $-1 + 0 - 1 + 0 \geq 0$.

e) Not a scalar product, definition requires $V \times V \to \mathbb{R}$ but $R_3 \neq R_2$.

f) Can be restated to

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right) \mapsto \begin{pmatrix} \langle (x_1, x_2), (y_1, y_2) \rangle \\ \langle (x_1, x_3), (y_1, y_3) \rangle \end{pmatrix}$$

and is thus a scalar product. (No, contradiction.)

Exercise 4

a) To show that a set of vectors form an orthogonal basis we show that they are a basis (they are linearly independent, done?) and that they are pairwise orthogonal (trivial).

b) Since the system

$$\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & \text{-1} & 0 & 1 \\ 1 & 1 & \text{-1} & 6 \\ 1 & \text{-1} & 0 & 3 \end{array}$$

does not have a solution, the given vector is not in U. (Cannot be constructed from linear combination of vectors in U.)

Exercise 5

$i = 1$

$$w_1 = u$$

$i = 2$

$$w_2 = v - (\text{proj}_{w_1}(v))$$

with

$$\text{proj}_{w_1}(v) = \frac{v \cdot w_1}{w_1 \cdot w_1} \cdot w_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

thus

$$w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

$i = 3$

$$w_3 = w - (\text{proj}_{w_1}(w) + \text{proj}_{w_2}(w))$$

with

$$\text{proj}_{w_1}(w) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$
$$\text{proj}_{w_2}(w) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

thus

$$w_3 = \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{pmatrix}.$$

Exercise 6

(i)

$$x_1^2 + x_1x_2 + x_2x_1 + 2x_2^2 - x_1x_3 - x_3x_1 + 3x_3^2 \geq 0$$
$$x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1x_3 + 3x_3^2 \geq 0$$
$$x_1^2 + 2x_2^2 + 3x_3^2 \geq -2x_1x_2 + 2x_1x_3$$

(ii)

$$x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 - x_1y_3 - x_3y_1 + 3x_3y_3 = y_1x_1 + y_1x_2 + y_2x - 1 + 2y_2x_2 - y_1x_3 - y_3x_1 + 3y_3x_3$$
$$0 = 0$$

(iii)

$$(\lambda x_1 + \phi y_1)z_1 + (\lambda x_1 + \phi y_1)z_2 + (\lambda x_2 + \phi y_2)z_1 + 2(\lambda x_2 + \phi y_2)z_2 - (\lambda x_1 + \phi y_1)z_3 - (\lambda x_3 + \phi y_3)z_1 + 3(\lambda x_3 + \phi y_3)z_3$$
$$= \lambda(x_1z_1 + x_1z_2 + x_2z_1 + 2x_2z_2 - x_1z_3 - x_3z_1 + 3x_3z_3) + \phi(y_1z_1 + y_1z_2 + y_2z_1 + 2y_2z_2 - y_1z_3 - y_3z_1 + 3y_3z_3)$$

Exercise 7 The angle can be calculated with

$$\cos(\alpha) = \frac{\langle u, v\rangle}{||u|| \cdot ||v||} = \frac{1}{\sqrt{\left(\sqrt{2}+1\right)^2 - 2\sqrt{2} + 1} \cdot 1}$$
$$\alpha = 60 \deg$$

Exercise 8 Gram-Schmidt auf Standardbasis von R3 anwenden.