Exercise 1

a) Linear.

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$$

$$f\left(\begin{pmatrix}\lambda u_1 + \mu v_1 \\ \lambda u_2 + \mu v_2 \\ \lambda u_3 + \mu v_3\end{pmatrix}\right) = \begin{pmatrix}\lambda(u_1 + 3u_2 + 4u_3) \\ \lambda u_3\end{pmatrix} + \begin{pmatrix}\mu(v_1 + 3v_2 + 4v_3) \\ \mu v_3\end{pmatrix}$$

$$\begin{pmatrix}\lambda u_1 + \mu v_1 + 3(\lambda u_2 + \mu v_2) + 4(\lambda u_3 + \mu v_3) \\ \lambda u_3 + \mu v_3\end{pmatrix} = \begin{pmatrix}\lambda(u_1 + 3u_2 + 4u_3) + \mu(v_1 + 3v_2 + 4v_3) \\ \lambda u_3 + \mu v_3\end{pmatrix}$$

b) Not linear. The first equation produces $u_1 v_2$ and $v_1 u_2$ in the first row, which has no chance of happening in the second equation; they are not equal.

$$f(u + v) = f\left(\begin{pmatrix}u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3\end{pmatrix}\right) = \begin{pmatrix}u_1 + v_1 + 2(u_2 + v_2) + (u_1 + v_1)(u_2 + v_2) + u_3 + v_3 \\ u_1 + v_1 + u_2 + v_2 + u_3 + v_3\end{pmatrix}$$

$$\begin{aligned} f(u) + f(v) &= \begin{pmatrix}u_1 + 2u_2 + u_1 u_2 + u_3 \\ u_1 + u_2 + u_3\end{pmatrix} + \begin{pmatrix}v_1 + 2v_2 + v_1 v_2 + v_3 \\ v_1 + v_2 + v_3\end{pmatrix} \\ &= \begin{pmatrix}u_1 + 2u_2 + u_1 u_2 + u_3 + v_1 + 2v_2 + v_1 v_2 + v_3 \\ u_1 + u_2 + u_3 + v_1 + v_2 + v_3\end{pmatrix} \end{aligned}$$

c) Not linear because $\lambda f(u) + \mu f(v)$ will necessarily have $\lambda + \mu$ in it. The statements are not equal.

$$f(\lambda u + \mu v) = f\left(\begin{pmatrix}\lambda u_1 + \mu v_1 \\ \lambda u_2 + \mu v_2 \\ \lambda u_3 + \mu v_3\end{pmatrix}\right) = \lambda u_1 + \mu v_1 + \lambda u_2 + \mu v_2 + \lambda u_3 + \mu v_3 + 1$$

d) Not linear.

$$f(\lambda u + \mu v) = f\left(\begin{pmatrix}\lambda u_1 + \mu v_1 \\ \lambda u_2 + \mu v_2\end{pmatrix}\right) = \begin{pmatrix}\lambda u_2 + \mu v_2 \\ \lambda u_1 + \mu v_1\end{pmatrix} = \lambda \begin{pmatrix}u_2 \\ u_1\end{pmatrix} + \mu \begin{pmatrix}v_2 \\ v_1\end{pmatrix} \neq \lambda \begin{pmatrix}u_1 \\ u_2\end{pmatrix} + \mu \begin{pmatrix}v_1 \\ v_2\end{pmatrix}$$

Exercise 3

a)

$$\begin{pmatrix}1 \\ -1\end{pmatrix}$$

b)

Exercise 4

a) The vectors in both B and C are linearly independent and thus both form a base of R^2.

$$\left.\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right. \quad \left.\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right. \quad \text{and} \quad \left.\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & -2 & 0 \end{array}\right. \quad \left.\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -3.5 & 0 \end{array}\right.$$

b) Since

$$\left.\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & -2 & 1 \end{array}\right. \quad \left.\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \end{array}\right. \quad \text{with } x_1 = \frac{5}{7} \text{ and } x_2 = -\frac{1}{7}$$

$$\left.\begin{array}{cc|c} 2 & 3 & -1 \\ 1 & -2 & 1 \end{array}\right. \quad \left.\begin{array}{cc|c} 2 & 3 & -1 \\ 0 & -\frac{7}{2} & \frac{3}{2} \end{array}\right. \quad \text{with } x_1 = \frac{1}{7} \text{ and } x_2 = -\frac{3}{7}$$

we have

$$A_C^B = \begin{pmatrix} \frac{5}{7} & \frac{1}{7} \\ -\frac{1}{7} & -\frac{3}{7} \end{pmatrix}.$$

c)

$$\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{5}{2} \end{pmatrix}$$

d)

$$\left.\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 1 & 1 \end{array}\right. \quad \text{and} \quad \left.\begin{array}{cc|c} 1 & -1 & 3 \\ 1 & 1 & -2 \end{array}\right.$$

$$\begin{pmatrix} \frac{1}{7} & -\frac{3}{7} \\ \frac{1}{2} & -\frac{5}{2} \end{pmatrix}$$